PERGAMON

International Journal of Solids and Structures 37 (2000) 2915–2935

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

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A multidimensional anisotropic strength criterion based on Kelvin modes

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Received 12 May 1998; in revised form 19 November 1998

Abstract

A new theory for the prediction of multiaxial strength of anistropic elastoplastic materials is proposed. The resulting failure envelope, in a multidimensional stress space, is piecewise smooth. Each facet of the envelope is expected to represent the locus of failure data by a particular anisotropic elastic deformation mode called a Kelvin mode. It is shown that the Kelvin mode theory alone provides an incomplete description of the failure of some materials, but that this weakness can be addressed by the introduction of a set of complementary modes. A revised theory which combines both Kelvin and complementary modes is developed and illustrated by an applied example. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

Material failure theories are used by engineers to extend the usefulness of a few strength test results to cover situations not tested. Isotropic failure theories have been available for over a century. Though none of these theories have been generalized, engineers have succeeded in matching many common materials with an applicable failure theory. Unlike their isotropic counterparts, anisotropic materials exhibit directionally dependent mechanical properties and thus the isotropic strength theories are inapplicable for them. Out of necessity, several empirical criteria have been developed for the prediction of strength of these new materials (Rowlands, 1985). The principal motivation for these criteria has been the prediction of strength of laminates of composite materials. Thus almost all of these anisotropic

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PII: S0020-7683(98)00338-2

criteria are planar and more appropriately applied to two-dimensional problems. Increasingly complex designs require new strength criteria which are not limited to biaxial stress conditions. Biomechanical engineers have long been aware of the anisotropic nature of the mechanical properties of organic materials. Wood and bone are materials which exhibit such a directionally dependent strength and stiffness. The stress states that these materials may be subjected to are generally multiaxial rather than biaxial. Some of the currently available two-dimensional strength criteria are theoretically extendable to stress states other than biaxial, but present difficulties both in their implementation and limitations in their results when such an extension is attempted. With few exceptions these criteria are simply various implementations of empirical correlation curves. The complete determination of these correlations becomes unusually difficult and impractical when applied to multiaxial stress. Furthermore, because these criteria are empirical, their strength predictions are, at best, only interpolative. With one exception, none of these multiaxial anisotropic strength criteria can be considered genuine theories. The object of this work is to present a candidate for a multiaxial anisotropic failure theory and to suggest that this candidate is a genuine theory and a viable alternative to the existing failure criteria for anisotropic materials.

The principles for the development of eigentensors of linear anisotropic elasticity were first introduced in 1856 by Lord Kelvin (Thompson, 1856) and rediscovered by Rychlewski (1984) and by Mehrabadi and Cowin (1990). Extending the concepts associated with isotropic elastic materials, namely that stress and strain may be decomposed into their dilatational and deviatoric parts, Mehrabadi and Cowin (1990) state the following general properties for anisotropic eigentensors:

(A) For any elastic symmetry the total stress tensor and the total strain tensor can be additively decomposed into a sum of six or fewer eigentensors.
(B) For any elastic symmetry each eigenstress is directly proportional to associated eigenstrain.
(C) For any elastic symmetry there is an additive decomposition of the total strain energy into a sum of six or fewer terms, each term being a scalar-valued product of the stress and strain eigentensors. These terms represent energy modes which are not interactive.

Property C allowed Biegler and Mehrabadi (1993) to suggest an extension of the von Mises plasticity theory to anisotropic materials. Their study shows how a strength criterion may be developed for anisotropic elastoplastic materials by postulating that failure would occur when any of the dilatational or deviatoric parts of the strain energy would exceed a critical value. Unlike the Tsai–Wu and its variants which restrict the shape of the failure envelope to that of an ellipsoid, this criterion would exhibit modes of failure or Kelvin modes. The failure envelope, in this case, would be piecewise smooth in a multidimensional stress space. Thus polyhedral envelopes of various shapes and aspect ratios are possible. Each facet of the envelope would represent the locus of failure data by a specific Kelvin mode. The premise this theory is based on is that failure occurs when the energy in any of the Kelvin modes has reached a critical value. The failure envelope which results from this premise is therefore an explicit predictor of strength whereas an empirical correlation such as the Tsai–Wu can only offer an implicit one.

2. Isotropic materials and decomposition of strain energy

A good summary of the commonly accepted failure criteria for isotropic materials can be found in Rowlands (1985). With few exceptions, none of these isotropic criteria can be extended to anisotropy. In this study however, an extension of the von Mises theory will be presented so that a review of this relatively successful isotropic theory is appropriate.

Though the *maximum strain energy* is sometimes mentioned in the literature it has been superseded by the more accurate *maximum distortional energy* theory. Huber is usually credited for proposing this theory (Huber, 1904), though Maxwell had suggested it years earlier in his letter to William Thomson on 18 December 1856 (Larmor, 1937). This theory uses the fact that the total strain energy exerted on an isotropic material is the sum of the *distortional* and *dilatational* energies. The theory postulates that the energy associated with the distortion of a material under triaxial stress determines the onset of yielding, or, in terms of the principal stresses, that,

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=2\sigma_{\mathrm{yp}}^2 \tag{1}$$

where σ_{yp} is the uniaxial tensile yield stress.

This failure envelope, in a principal stress space, is a tube of infinite length with its axis oriented along the median vector (Fig. 1(b)). The fact that this failure tube is open-ended is often ignored since the hydrostatic yield for many materials is quite high (Bridgeman, 1923; Nadai, 1950). A complete three-dimensional failure theory for isotropic materials would include two yield criteria, one for a distortional mode of failure (maximum von Mises) and one for a dilatational mode such as a maximum hydrostatic stress. The dilatational mode failure surface would be that of a plane normal to the axis of the von Mises tube and capping each end of it. As a consequence of this theory a complementary dilatational criterion is applied for triaxial stress, that is,

$$\frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)=\sigma_{\mathrm{H}}, \tag{2}$$

where σ_{H} represents the hydrostatic compressive or tensile failure stress which may be of different magnitudes.

The *maximum octahedral* or *von Mises* criterion (von Mises, 1928; Hencky, 1924) predicts that the onset of yielding occurs when the shear stress acting on an octahedral plane has reached a prescribed limit. It can be demonstrated that this criterion produces the same failure envelope as the *maximum distortional strain energy* criterion and is related to the latter by

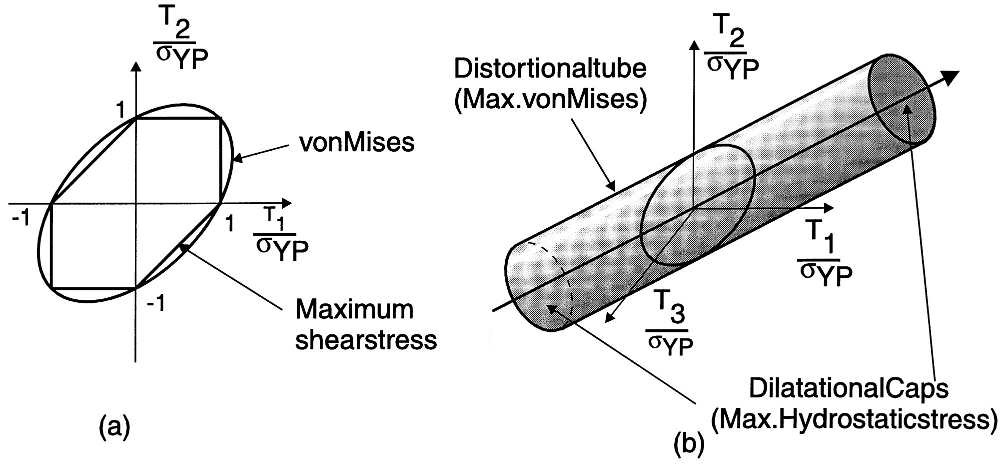

Fig. 1. Illustration of the three-dimensional distortional energy envelop along with the dilatational ends (b).The maximum shear stress is sometimes used as an alternative to the von Mises criterion (a).

$$U_{\text{distortional}} = \frac{3}{2}\frac{1+\nu}{E}\tau_{\text{oct}}^2. \tag{3}$$

This theory (along with maximum distortional energy) is widely accepted as most suitable in predicting the failure of isotropic ductile materials under multiaxial stress. However it should be noted that the formulations of these theories do not allow for distinct compression and tensile failure strengths and are therefore unsuitable for materials with this characteristic. Other theories have been proposed which can accommodate this difference such as *maximum principal stress* (*Rankine's theory*), or *Coulomb–Mohr's theory* (see, e.g. Rowlands, 1985). The popularity of some of these theories is often less dependent upon their accuracy than on their convenience and conservative predictions. For instance, because of the succinctness of its formulation, the *maximum shear stress* criterion is sometimes used as an alternative to the von Mises (Fig. 1(a)) even though experimental testing suggests that it may underestimate the strengths in biaxial stress states. Coulomb's theory is also often used to predict the strengths of aggregates (concrete, soils, sands, etc.), but it also may underestimate the strengths in biaxial stress e.g. concrete (Kupfer and Gerstle, 1973) sands (Green and Bishop, 1969) clays (Lade and Musante, 1978). For those materials that do not fit the general case, empirical curves have been formulated to accommodate their particularities (Podgórski, 1983).

3. Failure criteria for anisotropic materials

The failure theories described in the previous section are restricted to elastically isotropic materials. However, composites such as laminates, fibrous materials such as textiles and woods and biological materials such as bone, ligaments/tendons and skin all share the characteristic that their strength and stiffness are directionally dependent. These generally anisotropic materials display more complex interaction of multiaxial stresses and strains, making the development of reliable failure theories more difficult.

A thorough review of the anisotropic strength theories which have been proposed up to 1985 is provided by Rowlands (1985). There are several theories that have been used with some success. These usually assume that the failure envelope in a stress or strain space is a single function. The Tsai–Hill theory, for instance, assumes the multiaxial failure is bound by a closed quadratic surface: an ellipsoid. The failure bound in terms of the stresses T_i is described by

$$F_{ij}T_iT_j = 1, \tag{4}$$

where $i, j = 1, \ldots, 6$ and F_{ij} are coefficients determined from mechanical tests (Hill, 1950). The Tsai–Wu failure criterion is a quadric of the form

$$F_{ij}T_iT_j + F_iT_i = 1, \tag{5}$$

where $i, j = 1, \ldots, 6$, F_i and F_{ij} are the vector and second rank six-dimensional strength tensors, the coefficients of which are determined from mechanical tests (Tsai and Wu, 1971). This theory permits different compressive and tensile strengths. Most of the other Tsai–Wu type theories suggested so far are variations of the two basic forms, eqns (4) and (5). Some, like Cowin (1979) and Hoffman (1967), provide predictions of the F_{ij} and F_i coefficients. Others, like Tennyson et al. (1980) and Wu and Scheublein (1974), extend the Tsai–Wu criterion to include additional cubic terms.

Although the Tsai–Wu failure criterion and its variants have found considerable acceptance in the literature, it appears that this is mainly based on their ability to account for biaxial strength and the dissymmetry in the compressive and tensile strengths. The Tsai–Wu theory has been applied to biaxial data with tolerable success but not to triaxial data. Close examination of the tensor polynomial (5) and

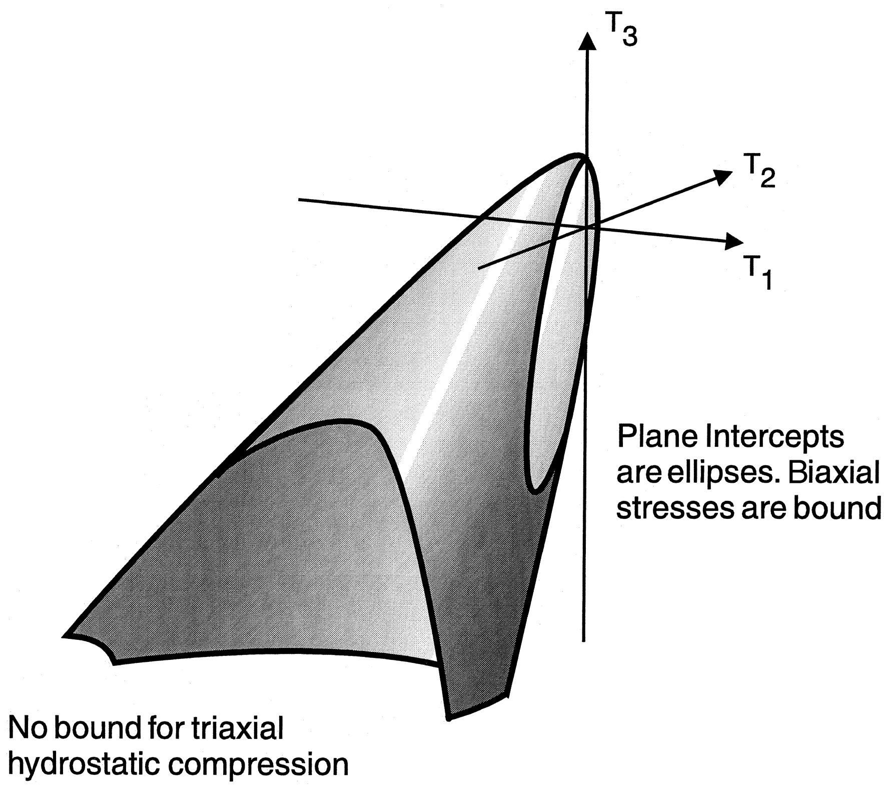

Fig. 2. Example of a limitation for the model presented by Tsai–Wu for a failure criterion. This surface is a three-dimensional representation of the Tsai–Wu criterion (7) for bovine cortical bone provided by Cowin (1989). This plot, in principal stress space, is unbounded for triaxial compressive stress. Even though the criterion satisfies the restriction specified by Tsai and Wu (1971), it provides an incomplete description of the three-dimensional failure envelop.

the constraints on the stress interaction terms F_{ij}, suggests that the model presented by Tsai and Wu is more apt at fitting the biaxial stress failure data than it is in providing a full three-dimensional failure envelope. Indeed Tsai and Wu (1971) prescribe that the diagonal terms of the strength tensor **F** be positive and that the off-diagonal terms satisfy a stability condition. That is to say that,

$$F_{ii} > 0, \quad \text{and} \quad \begin{vmatrix} F_{ii} & F_{ij} \\ F_{ij} & F_{jj} \end{vmatrix} > 0 \quad \text{for} \quad i, j = 1 \ldots 6. \tag{6}$$

Unfortunately, Tsai and Wu (1971) mistakenly represent these conditions as sufficient to guarantee that "...the shape of the surface will not be open-ended like a hyperboloid." This is not, in fact, the case. The requirement that the failure envelope be completely bound is a stronger restriction than that imposed by eqn (6). Indeed the stability condition (6) does not guarantee that the strength tensor **F** be positive definite, only that its one rowed and two rowed minors are. Thus, the resulting quadric is guaranteed to produce closed bounds in three principal biaxial stress spaces, but not necessarily in a triaxial stress space. The Tsai–Wu criterion of Fig. 2 is an example of such a quadratic surface. A three-dimensional Tsai–Wu strength criterion for bovine cortical bone is provided by Cowin (1989) as

$$\begin{aligned} &20.38T_1 + 13.33T_2 + 2.2T_3 + 135T_1^2 + 133.3T_2^2 + 27T_3^2 + 322T_6^2 + 229.3T_5^2 + 200.5T_4^2 \\ &\quad - 2(26.9)T_1T_2 - 2(54.4)T_1T_3 - 2(39.2)T_2T_3 = 1, \end{aligned} \tag{7}$$

where the units of the stress matrix components are GPa. This strength criterion satisfies all the conditions imposed by the Tsai–Wu criterion but is not an ellipsoid. Though this surface does intercept the biaxial planes in such a way as to produce elliptical curves, it does not exhibit a bound for triaxial compression and therefore would be unsuitable to describe any three-dimensional strength. Therefore, in a principal stress space, the Tsai–Wu criterion, eqn (5), with the restriction, eqn (6), is not a true three-dimensional criterion, but in fact, a set of three two-dimensional criteria. Each two-dimensional Tsai–Wu ellipse may be independently fitted to biaxial data by adjusting the appropriate stress interaction terms F_{12}, F_{13} or F_{23}. But in order to guarantee that the strength data are fitted to an ellipsoid, all three stress interaction terms would have to be adjusted simultaneously while verifying that the strength tensor is positive definite for each adjustment. This would be a very difficult task to accomplish and even if successful, it is unlikely that the resulting failure surface could fit the biaxial data as well as eqn (5) with eqn (6).

In 1974, Wu and Scheublein suggested that the principal directions of strength need not be orthogonal and hence the failure surface is not necessarily an ellipsoid (Wu and Scheublein, 1974). They thus suggest that third-order terms be included in the tensor polynomial (5) so that the failure envelope becomes

$$F_iT_i + F_{ij}T_iT_j + F_{ijk}T_iT_jT_k = 1 \tag{8}$$

This extension to the theory allows the possibility that a single function describes a closed strength envelope that is not an ellipsoid such as the one described by eqn (5). However, the determination of the third order constants of eqn (8) would require several triaxial and combined tensile and shear tests. These tests are extremely difficult to accomplish accurately and therefore limit the usefulness of such a theory.

For the past quarter century the Tsai–Wu criterion has been a popular choice for predicting the biaxial strength of planar components such as laminates. This popularity is due in part to the conciseness of its formulation and because it produces simple convex bounds. The tradeoff of such conciseness however is that extending the Tsai–Wu beyond biaxial stress is not easily accomplished. Theories such as the Tsai–Wu variants produce failure envelopes that are described by a single continuous function. In practice, however, many materials exhibit multiple failure mechanisms so that the failure data may not be smoothly distributed. An appropriate single function would be either too complex to be determined or too simple to accurately describe the failure data distribution. This has led some investigators to believe that an appropriate failure envelope for anisotropic materials should not be produced from a single function, but rather from a set of independent functions. Each of these functions would describe the failure of a material by a particular mode of failure.

4. Modal failure criteria

In an attempt to correlate a failure mechanism, or a failure mode, with the failure envelope, some investigators have introduced modal failure theories (Hashin, 1980; Feng, 1991; Biegler and Mehrabadi, 1993). The premise of such theories is that anisotropic materials may fail by multiple independent mechanisms; therefore, their failure bounds in a stress or strain space are not required to be smooth, but simply piecewise smooth (Hashin, 1980). The failure envelopes for such materials would therefore be a composite of patches. Each patch represents the locus of failure by a particular mode. For instance the *maximum strain* theory predicts that failure will occur in a unidirectionally reinforced composite when either the maximum axial, transverse or shear strain is exceeded. The failure envelope of a laminate composed of multiple layers of uniaxial plies may be produced by superimposing the individual ply predictions. Practical application of such a theory, however, is limited to biaxial stress or strain.

In order to correlate a failure mechanism with the failure envelope, Hashin (1980) introduced a mechanism-based failure criterion for transversely isotropic elastoplastic materials. Hashin (1980) assumes a criterion which is a quadratic polynomial of the stress invariants. Then by selectively eliminating some stress interaction terms, Hashin (1980) observes that the remaining stress terms may be segregated into two groups: those pertinent to a fiber failure plane and others to a matrix failure plane. As a result, the failure criteria can be decoupled into two modal criteria: a matrix and a fiber mode. Hashin (1980) further argues that since the failure mechanisms are different for tension and compression, the fiber and matrix modes must be solved for compression and tension separately. This produces a total of four surfaces, the constants of which must be determined from experiment. This represents an appealing approach. Unlike the Tsai–Wu criterion which produces a failure envelope which is a single smooth surface, this approach results in a much more flexible, piecewise continuous failure envelope.

Unfortunately, the method suffers from several shortcomings. These shortcomings are discussed in more detail in Arramon (1997) but are summarized here. Firstly, the arguments used to decouple the failure criteria into fiber and matrix modes are only applicable for transversely isotropic materials. It is difficult to see how this same procedure can be applied to any other type of material symmetry. Secondly, the theory apparently produces a fiber mode which must always reduce to a maximum principal stress criterion. This prediction implies that a unidirectional fiber-reinforced lamina subjected to biaxial stress should fail by maximum principal stress. This is not confirmed by experiment. Finally, the boron epoxy example used by Hashin (1980) to illustrate the application of this strength criterion failed to exhibit a bound in transverse biaxial compression (matrix mode). The resulting strength envelope for this material is thus an inappropriate multiaxial strength predictor.

In 1991, Feng (1991) used a similar approach to suggest a strain invariant-based failure criterion. Though the author did not plot the failure surface in a principal stress space, it is clear that this procedure should, with the same failure data, produce a similar failure surface as in Hashin and therefore would exhibit the same weaknesses.

The criteria for these modes of failure are selected because their formulation only involves macroscopic stresses significant to a specific microscopic interaction. Other three-dimensional failure envelopes such as those produced by Biegler and Mehrabadi (1993) and Schreyer and Zuo (1995) are based on the premise that the symmetry of the failure envelope is related to that of the elasticity tensor.

5. Failure criteria based on Kelvin modes

Using the foundation provided by Mehrabadi and Cowin (1990), Biegler and Mehrabadi (1993, 1995) present a generalization of the von Mises plasticity theory for anisotropic materials. In a later study, Schreyer and Zuo (1995) used the same spectral decomposition of the elasticity tensor described by Biegler and Mehrabadi (1993) to produce yield surfaces based on projection operators. It can be shown that the projection operators described by Schreyer and Zuo (1995) are the modal tensors described by Biegler and Mehrabadi (1993, 1995) and again by Martin (1994). Thus both approaches result in the same yield surfaces. The formulation for these failure criteria is based upon using the spectral decomposition of the six-dimensional elasticity tensor $C_{\alpha\beta}$ such that

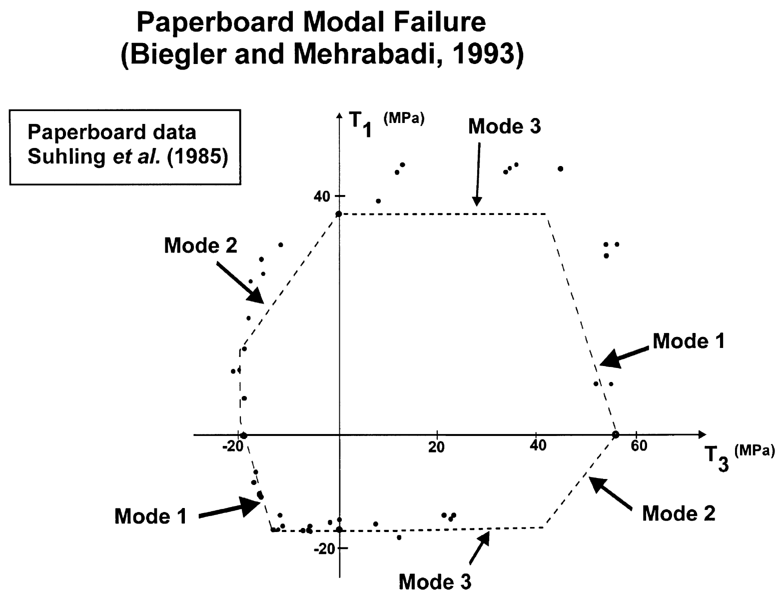

Fig. 3. Plot of the modal failure envelope as described by Biegler and Mehrabadi (1993) and Martin (1994). The experimental failure data for paperboard (Suhling et al., 1985) is superposed. The quadrants of the principal stress plane define regions where specific modes are (or are not) active. Notice the fit of the corner in the biaxial compression region. The experimental data suggest that this corner does indeed exist and that is formed by the inersection of two virtually rectilinear bounds.

$$C_{\alpha\beta} = \sum_{A=1}^{K} \Lambda_A P_{\alpha\beta}^{(A)}, \tag{9}$$

where $K(K \leqslant 6)$ is the number of Kelvin modes i.e. the number of distinct eigenvalues Λ_A and $P_{\alpha\beta}^{(A)}$ are the modal tensors (Mehrabadi and Cowin, 1990). The number of distinct eigenvalues governs the number of modal tensors $P_{\alpha\beta}^{(A)}$ for a given symmetry.

The stresses T_α and strains E_α may thus be decomposed into K eigenstresses $T_\alpha^{(A)}$ and eigenstrains $E_\alpha^{(A)}$ such that

$$T_\alpha^{(A)} = P_{\alpha\beta}^{(A)} T_\beta,$$

$$E_\alpha^{(A)} = \frac{1}{\Lambda_A} T_\alpha^{(A)}, \quad (\alpha, \beta = 1, \ldots, 6, \quad A = 1, \ldots, K). \tag{10}$$

The strain energy U may therefore also be decomposed into the sum of *modal* energies $U^{(A)}$ so that

$$2U^{(A)} = T_\alpha^{(A)} E_\alpha^{(A)} = \frac{1}{\Lambda_A} T_\alpha^{(A)} T_\alpha^{(A)} = \frac{1}{\Lambda_A} P_{\alpha\beta}^{(A)} T_\beta P_{\alpha\gamma}^{(A)} T_\gamma = \frac{1}{\Lambda_A} P_{\alpha\beta}^{(A)} T_\alpha T_\beta, \quad (\alpha = 1, \ldots, 6, \quad A = 1, \ldots, K) \tag{11}$$

Once these energy modes have been defined for a given material, the threshold values are determined from uniaxial tests. Failure is predicted to occur when the energy stored in any one of these domains reaches a critical value. Both Biegler and Mehrabadi (1993) and Schreyer and Zuo (1995) point out that the method produces failure bounds in a manner completely analogous to von Mises plasticity theory. In fact the isotropic case for this modal failure theory produces two distinct eigenvalues and thus two modes of failure: a dilatational mode (or a maximum hydrostatic stress failure) and an isochoric shear mode (a maximum distortional energy or maximum octahedral or von Mises stress).

A comparison of the experimental data for paperboard (Suhling et al., 1985) and the predictions of the modal failure analysis for tetragonal symmetry was provided by Biegler and Mehrabadi (1993) and later again by Martin (1994). The procedure employed in these two studies uses a protocol of active regions to fit the different compressive and tensile strengths to the modal lines illustrated in Fig. 3. The premise of this active regions protocol is that the quadrants of principal stress determine whether a particular mode is active and must be considered or is not and can be ignored. The resulting 2D failure envelope is intuitively appealing. Though the failure bounds differ somewhat from the experimental data in the biaxial tensile domain, they fit particularly well in the biaxial compressive region. The modes clearly describe the corner which appears in the biaxial compression experimental data. Not only does this corner lend credibility to the concept of modes of failure but also to the premise that these rectilinear bounds are produced from the intercept of the modal failure planes with this biaxial stress plane. The active regions protocol cannot be extended to three dimensions. A revised three-dimensional protocol is proposed in the following sections. This new protocol uses the equivalence between surfaces of modal energies described in this section and surfaces of constant eigenstress magnitude to produce a new criterion, one which can easily be extended to a multi-dimensional stress space.

6. A new modal failure theory

There are two problems associated with applying the Kelvin mode failure criterion discussed in the previous section. First, the active regions procedure employed by Biegler and Mehrabadi (1993) and later again by Martin (1994) to fit the paperboard failure data of Suhling et al. (1985) proved to be inapplicable in a three-dimensional stress space. Second, the criterion fails to account for different compressive and tensile strength for an open failure envelope in the three-dimensional stress space when biaxial stress data are used. Hence, to overcome these problems, two revisions are made as follows:

(a) The modal energies and the active regions protocol employed by Biegler and Mehrabadi (1993, 1995) are replaced by maximum eigenstress magnitudes evaluated in positive and negative senses. The resulting maximum eigenstress magnitudes are completely equivalent to the maximum modal energies. The eigenstress senses provide a means of accommodating different compressive and tensile strengths in a multi-dimensional principal stress space.

(b) A set of complementary modes are introduced to accommodate any remaining strength data which cannot be attributed to either a distortional or dilatational mode of failure.

6.1. Eigenstress magnitudes as a representation of modal energies

The intent of the active regions protocol described earlier was to accommodate the different tensile and compressive strengths of paperboard. An alternative protocol may be formulated by observing that the surfaces of modal energy are also surfaces of constant eigenstress. Suppose Λ is distinct (multiplicity of 1), then eqn can be rewritten as be rewritten as

$$2\Lambda_A U^{(A)} = P_{\alpha\beta}^{(A)} T_\alpha T_\beta = T_\alpha^{(A)} T_\alpha^{(A)} = \left(\sigma^{(A)}\right)^2, \quad (\alpha = 1, \ldots, 6, \quad A = 1, \ldots, K) \tag{12}$$

where the scalar $\sigma^{(A)}$ is the magnitude of the eigenstress $T_\alpha^{(A)}$. In the case of isotropy, when A is taken to

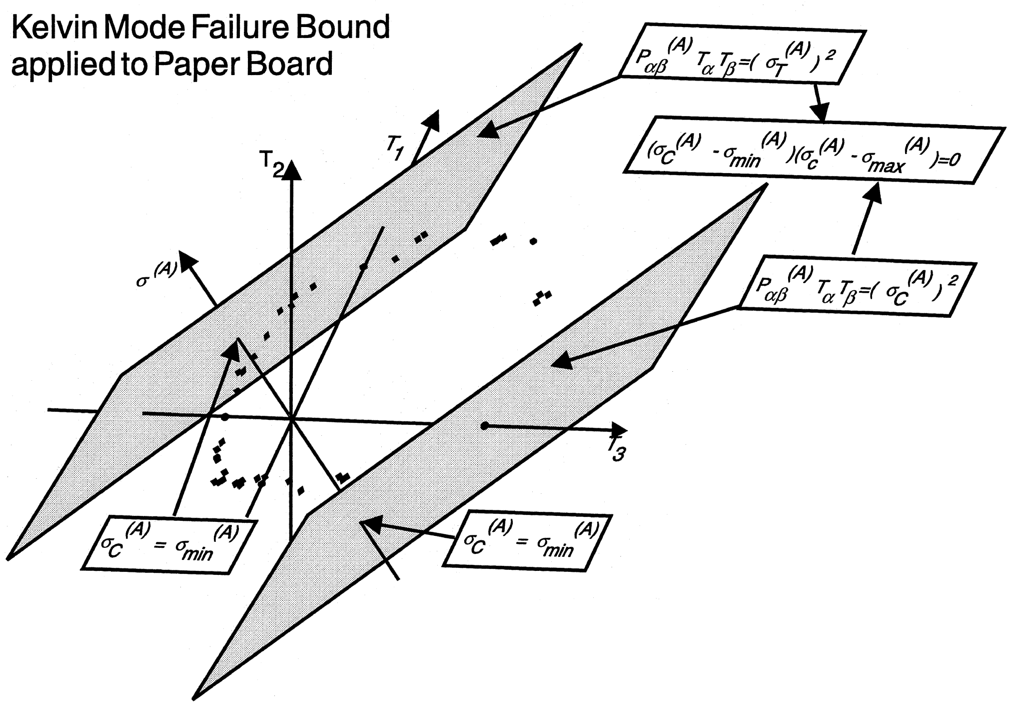

Fig. 4. Illustration of how the compressive and tensile modal bounds are determined. The failure data plotted here are the same paperboard biaxial data as in Fig. 3. The two modal energy bounds $P_{\alpha\beta}^{(A)} T_\alpha T_\beta = (\sigma_{\min}^{(A)})^2$ and $P_{\alpha\beta}^{(A)} T_\alpha T_\beta = (\sigma_{\max}^{(A)})^2$ define maximum and minimum allowable eigenstress magnitudes. Note that, for clarity, although the bounds describes planes in a three-dimensional stress space, only the (T_2, T_3) intercept is shown here.

be the distortional mode, this equation reduces to the well known equation that relates the distortional energy to the square of the second invariant of the deviatoric stress.

To accommodate the dissymmetry in tensile and compressive failure modes, we presume that the maximum modal energies must be evaluated for positive and negative eigenstresses $\sigma^{(A)}$ separately. Thus we obtain two criteria for each mode:

$$2\Lambda_A U_T^{(A)} = \left(\sigma_T^{(A)}\right)^2 \quad \text{and} \quad 2\Lambda_A U_C^{(A)} = \left(\sigma_C^{(A)}\right)^2. \tag{13}$$

These conditions define the extreme bounds of the Kelvin mode (A) in stress space. Ideally, the tensile and compressive eigenstress magnitudes $\sigma_T^{(A)}$ and $\sigma_C^{(A)}$ should be determined from a triaxial test. In lieu of such tests, however, these bounds may be estimated from whatever failure data are available. In Fig. 4 the same biaxial paperboard data employed by Biegler and Mehrabadi (1993) is used to estimate the maximum eigenstress magnitudes in a triaxial principal stress space. The eigenstress magnitudes are calculated for each failure datum. The most positive and most negative eigenstress magnitudes $\sigma_{\max}^{(A)}$ and $\sigma_{\min}^{(A)}$ are the estimates of $\sigma_T^{(A)}$ and $\sigma_C^{(A)}$. With these bounds on the eigenstress magnitudes the strength envelope is produced from K-Kelvin modes in the form

$$\left(\sigma^{(A)} - \sigma_T^{(A)}\right)\left(\sigma^{(A)} - \sigma_C^{(A)}\right) = 0 \tag{14}$$

where $\sigma_T^{(A)} = \sigma_{\max}^{(A)}$ and $\sigma_C^{(A)} = \sigma_{\min}^{(A)}$, $A = 1, \ldots, K$ and K is the number of distinct eigenvalues. Fig. 4 shows

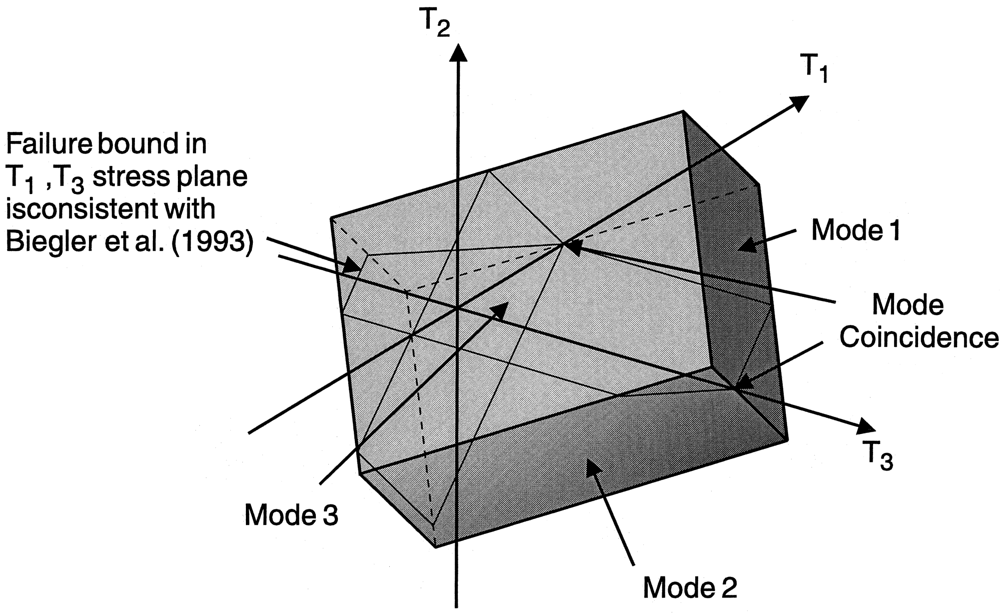

Fig. 5. A 3D plot of the Kelvin mode failure bounds for paperboard. The (T_1, T_3) planar intercept of this failure envelope coincides with the plots produced by Biegler and Mehrabadi (1993) and Martin (1994). Note that this plot was produced with the assumption that only the uniaxial data are available. For this reason modes 1 and 2 coincide on the T_3 axis and modes 2 and 3 on the T_1 axis. This would not necessarily be the case if the full set of biaxial were considered.

one such Kelvin mode superimposed upon the paperboard strength data which produced it. For the paperboard example of Biegler and Mehrabadi (1993) and of Martin (1994), this Kelvin mode criterion reduces to a set of three failure criteria given by

Kelvin mode 1

$$\left(\sigma^{(1)} - 47.1\right)\left(\sigma^{(1)} + 16.8\right) = 0, \quad \text{where} \quad \sigma^{(1)} = 0.382(T_1 + T_2) + 0.841T_3,$$

Kelvin mode 2

$$\left(\sigma^{(2)} - 18.1\right)\left(\sigma^{(2)} + 30.3\right) = 0, \quad \text{where} \quad \sigma^{(2)} = 0.595(T_1 + T_2) - 0.541T_3,$$

Kelvin mode 3

$$\left(\sigma^{(3)} - 21.6\right)\left(\sigma^{(3)} + 9.19\right) = 0, \quad \text{where} \quad \sigma^{(3)} = \frac{1}{\sqrt{2}}(T_1 - T_2). \tag{15}$$

We therefore obtain two planes for each of the three modes in eqn (15). The resulting failure envelope, illustrated in illustrated in Fig. 5, is a brick shaped volume. The normal to each face of the brick is a vector in the direction of the eigenstresses. The intercept of this volume with the (T_1, T_3) principal stress plane is entirely consistent with the two-dimensional predictions of Biegler and Mehrabadi (1993) and Martin (1994) illustrated in Fig. 3. The advantage that this approach presents is that it dispenses with the need for the active regions protocol previously mentioned. As in these studies, only the uniaxial test data were considered to produce Fig. 5. Consequently, some of the failure surfaces intersect on the axes (mode 1 and mode 2 on both the T_1 and T_3 axes of Fig. 5). It is possible but not

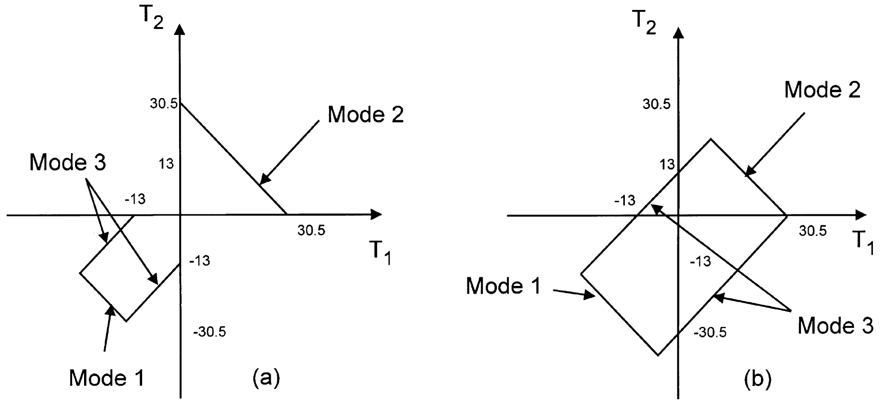

Fig. 6. Illustration of the problem with the (T_1, T_2) Kelvin mode predictions for paperboard strengths. (a) If the tensile/compressive strengths are considered equal, then discontinuities across the axes appear unavoidable. (b) If instead the surfaces are assumed continuous then we have the counterintuitive situation with T_1(comp.) = T_2(tensile) and T_1(tensile) = T_2(comp.).

likely that any one, experimentally determined, strength datum represents the simultaneous failure by two or more modes. This mode coincidence is an ambiguity since the failure datum in question cannot be attributed to a specific Kelvin mode. A similar situation arises in plasticity theories in which the yield surface is piecewise. In such theories, the plastic strain direction is somewhere between the two normals to the yield surfaces at the vertex (if the flow rule is associated).

6.2. Complementary failure modes

Ideally, the tensile and compressive eigenstresses should be determined directly from triaxial tests. However triaxial tests are extremely difficult to reproduce accurately so that such data are almost never available. Biaxial data, such as the paperboard data of Suhling et al. (1985), are more common and may be used to estimate the bounds on the eigenstresses. Whenever possible, the biaxial data from multiple planes should be used. In this case, the biaxial data provided by Suhling et al. (1985) are for one plane only (T_1, T_3). However, for the tetragonal symmetry of paperboard it may reasonably be assumed that the strength in the (T_2, T_3) plane is the same as in the (T_1, T_3) plane, so that the intercept of the Kelvin modes with these two stress planes should produce the same predictions. Yet there is no way to

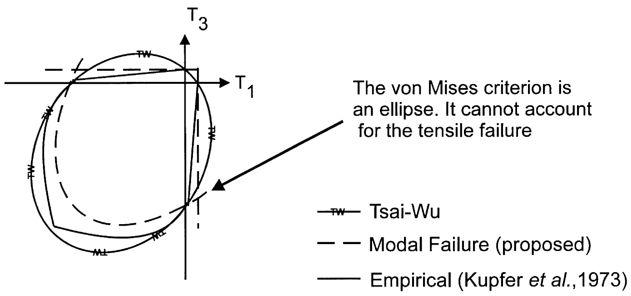

Fig. 7. Illustration of the von Mises theory's inability to account for the tensile strength of concrete. The empirical curve was produced from experiment (Kupfer and Gerstle, 1973). The addition of a complementary maximum principal stress mode produced a failure envelope consistent with the experimental data. The proposed combined criterion fits 'better' than the Tsai–Wu for this case.

accomplish this without introducing a gap in the failure envelope. A plot of the (T_1, T_2) Kelvin modes (Fig. 6) exhibits this flaw in the Kelvin mode criterion. If the tensile/compressive strengths are considered equal, then discontinuities across the axes appear unavoidable (Fig. 6(a)). If instead, the surfaces are assumed continuous then the counterintuitive situation arises with T_1(comp.) = T_2(tensile) and T_1(tensile) = T_2(comp.).

This problem appears unavoidable and is not unique to this material. Indeed, a similar inconsistency develops when this criterion is applied to elastically isotropic materials like concrete. For isotropy, this modal failure theory reduces the number of modes to two: a distortional (von Mises) and a dilatational (hydrostatic stress) mode. Though the von Mises agrees well with the biaxial compressive data of Kupfer and Gerstle (1973), neither the von Mises nor the maximum hydrostatic stress criteria can account for the failure in the tensile regions (Fig. 7). Clearly, concrete exhibits an additional mode of failure which is unrelated to the Kelvin modes. These aspects of the modal failure envelope of concrete and paperboard suggest that this criterion provides an incomplete description of the failure.

To accommodate the tensile strength of concrete, it is proposed to revise the Kelvin mode failure theory with the introduction of a set of complementary modes. A set of maximum principal stress modes (or (P) modes) has intuitive appeal and apparently addresses the weaknesses described above. These complementary modes would be determined in a manner analogous to the Kelvin modes, that is to say we define the (P) failure modes such that,

$$\left(\sigma^{(P)} - \sigma_T^{(P)}\right)\left(\sigma^{(P)} - \sigma_C^{(P)}\right) = 0, \tag{16}$$

where $\sigma^{(\mathrm{I})} = T_1$, $\sigma^{(\mathrm{II})} = T_2$ and $\sigma^{(\mathrm{III})} = T_3$. It is important to note that this criterion is not equivalent to a maximum uniaxial stress criterion. Instead, the bounds of $\sigma_T^{(P)}$ and $\sigma_C^{(P)}$ are evaluated from the multiaxial failure data so that $\sigma_T^{(P)} = \sigma_{\max}^{(P)}$ and $\sigma_C^{(P)} = \sigma_{\min}^{(P)}$.

When this complementary criterion is applied to the concrete failure data Fig. 7, it succeeds in accounting for the tensile strength of concrete. Indeed, an appropriate Tsai–Wu criterion could never account for the discontinuities apparent in the experimental data of Kupfer and Gerstle (1973).

6.3. Complementary failure modes

The revised modal theory may therefore be formally described by

$$\left(\sigma^{(A)} - \sigma_T^{(A)}\right)\left(\sigma^{(A)} - \sigma_C^{(A)}\right) = 0, \quad \text{and} \quad \left(\sigma^{(P)} - \sigma_T^{(P)}\right)\left(\sigma^{(P)} - \sigma_C^{(P)}\right) = 0, \tag{17}$$

where $\sigma^{(A)}$ is the magnitude of the eigenstress associated with the eigenvalue Λ_A of the material's elasticity tensor; $\sigma_T^{(A)} = \sigma_{\max}^{(A)}$ is the largest values of $\sigma^{(A)}$ that can be determined from the available failure data set; $\sigma_C^{(A)} = \sigma_{\min}^{(A)}$ is the smallest value of $\sigma^{(A)}$ that can be determined from the available failure data set; $A = 1, \ldots, K$ where K is the number of distinct eigenvalues of the material's elasticity tensor; $\sigma^{(P)}$ is the magnitude of the principal stress (complementary mode); $\sigma_T^{(P)} = \sigma_{\max}^{(P)}$ is the largest value of $\sigma^{(P)}$ that can be determined from the available failure data set; $\sigma_C^{(P)} = \sigma_{\min}^{(P)}$ is the smallest value of $\sigma^{(P)}$ that can be determined from the available failure data set.

7. Application of modal failure to paperboard

To illustrate the application of this theory, the paperboard data of Suhling et al. (1985) used by Biegler and Mehrabadi (1993) is reexamined. In this case, however, the entire set of biaxial stress data is employed to produce the mode bounds and the resulting envelope reproduced. Though this example lacks biaxial data in the transverse plane, the resulting envelope is almost completely defined and should provide sufficient illustration for this method.

There are two sets of data which are critical to the determination of the modal failure envelope for any material: the eigenvectors of the elasticity tensor and the bounds or critical values of the Kelvin and complementary modes. The Kelvin modes are planes of constant eigenstress. The orientations of these planes are dependant upon the eigenvectors of the material's elasticity tensor. It is critical therefore that these eigenvectors be determined as a first step in the analysis. The elastic constants for cardboard, as given by Martin (1994), are

$$\begin{array}{lll} E_1 = 3510\ \text{MPa}, & E_2 = 3510\ \text{MPa}, & E_3 = 6930\ \text{MPa}, \\ v_{13} = 0.15, & v_{23} = 0.15, & v_{12} = 0.3, \\ G_{23} = 1700\ \text{MPa}, & G_{13} = 1700\ \text{MPa}, & G_{12} = 1500\ \text{MPa}, \end{array}$$

which are in reasonable agreement with those listed in Suhling et al. (1989). The paperboard elasticity tensor is deduced in the usual manner from the elasticity data listed above and is given by

$$C = \begin{bmatrix} 4220 & 1520 & 1700 & 0 & 0 & 0 \\ 1520 & 4220 & 1700 & 0 & 0 & 0 \\ 1700 & 1700 & 7940 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1700 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1700 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1500 \end{bmatrix} \cdot \ \text{MPa}. \tag{18}$$

Cowin and Mehrabadi (1995b) provided a catalog of anisotropic symmetries along with the restriction on the elastic components. An examination of this elasticity tensor reveals that the material exhibits a tetragonal symmetry. The eigenvalues for this elasticity tensor are given as

$$\Lambda = \begin{bmatrix} 9480 \\ 4200 \\ 2700 \\ 1700 \\ 1700 \\ 1500 \end{bmatrix} \cdot \ \text{MPa}, \tag{19}$$

and the first three eigenvectors are

$$N1 = \begin{bmatrix} 0.382435 \\ 0.382435 \\ 0.841123 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad N2 = \begin{bmatrix} -0.594763 \\ -0.594763 \\ 0.540845 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad N3 = \begin{bmatrix} -0.7071068 \\ 0.7071068 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{20}$$

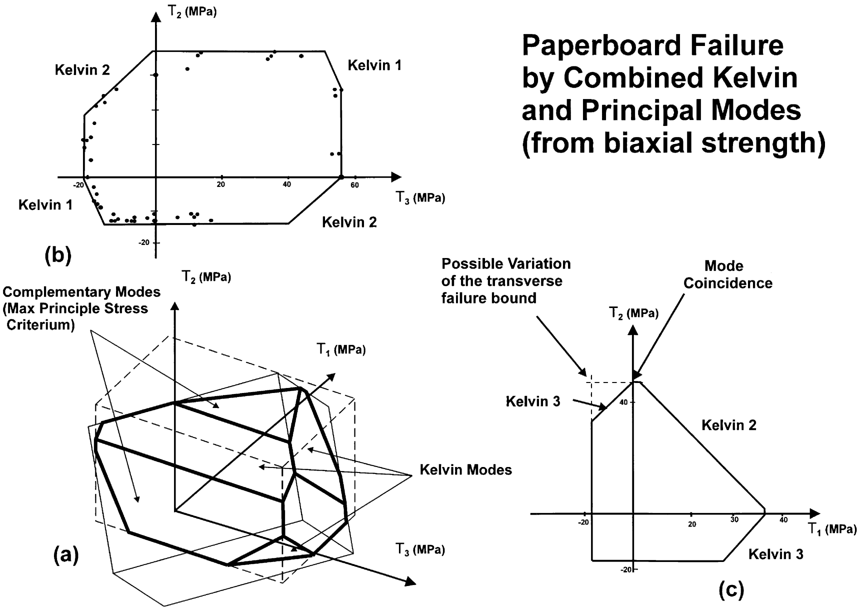

Fig. 8. Illustration of the revised three-dimensional modal failure envelope (a). The Kelvin and complementary modes are superimposed. The intersection of this mode sets is the proposed failure envelope. (b) The trace of the modal failure envelope in the (T_2, T_3) plane is consistent with the results proposed in Biegler and Mehrabadi (1993). Though not illustrated here the trace in (T_1, T_3) is identical to that in the (T_2, T_3) plane. (c) The trace of the modal failure envelope in the (T_1, T_2) plane. Transverse biaxial data are not available, therefore the (T_1, T_2) failure envelope may still display the variation shown in (c).

The last three eigenvectors are

$$N4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix}, \quad N5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix}, \quad N6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{21}$$

where θ is an arbitrary angle in the eigensubspace defined by $N5$. The first three modal products are then given by eqn (12) as

$$P^{(1)}_{\alpha\beta} T_\alpha T_\beta = (0.382(T_1 + T_2) + 0.841 T_3)^2 = (\sigma^{(1)})^2,$$

$$P^{(2)}_{\alpha\beta} T_\alpha T_\beta = (0.595(T_1 + T_2) - 0.541 T_3)^2 = (\sigma^{(2)})^2,$$

$$P^{(3)}_{\alpha\beta} T_\alpha T_\beta = (0.707 * (T_1 - T_2))^2 = (\sigma^{(3)})^2. \tag{22}$$

The last two modal products reduce to functions of the shears alone

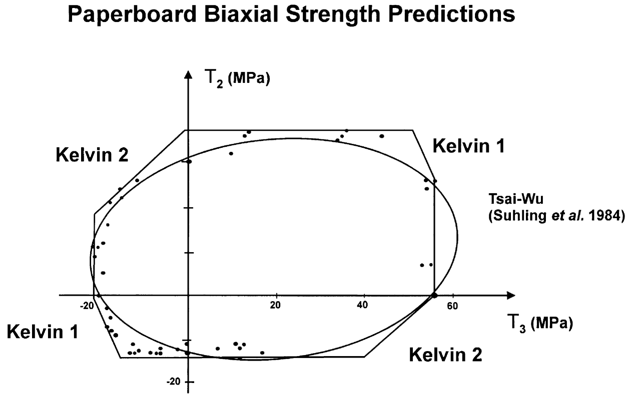

Fig. 9. Comparison of the biaxial modal failure predictions with an appropriate Tsai–Wu envelope (Suhling et al., 1985). The Tsai–Wu coeffcients are $F_1 = -4.4\text{E}-2\ \text{Mpa}^{-1}$, $F_{11} = 2.5\text{E}-3\ \text{Mpa}^{-2}$, $F_3 = -3.2E-2\ \text{Mpa}^{-1}$, $F_{33} = 8.9E-4\ \text{Mpa}^{-2}$ and the stress interaction term is determined using the relationship developed by Cowin (1979) that is: $F_{13} = -3.3\text{E}-4\ \text{Mpa}^{-2}$.

$$P^{(4)}_{\alpha\beta} T_\alpha T_\beta = (T_4)^2 + (T_5)^2 = \left(\sigma^{(4)}\right)^2,$$

$$P^{(5)}_{\alpha\beta} T_\alpha T_\beta = (T_6)^2 = \left(\sigma^{(6)}\right)^2. \tag{23}$$

These last two modes do not involve T_1, T_2, or T_3 and therefore have no meaningful representation in this principal stress space.

In their paperboard example Biegler and Mehrabadi (1993) only considered the uniaxial strengths. In this case however, the entire biaxial strength data are included in the evaluation of the modal failure envelope. The data were interpreted from the biaxial strength plot in Martin (1994) for the same paperboard data studied by Suhling et al. (1985). This strength data, along with the corresponding Kelvin mode magnitudes, are tabulated in Arramon (1. The maximum and minimum Kelvin mode magnitudes and principal stresses are identified. Then, with eqn (14), the three Kelvin modes become Kelvin mode 1

$$\left(\sigma^{(1)} - 47.1\right)\left(\sigma^{(1)} + 16.8\right) = 0, \quad \text{where } \sigma^{(1)} = (0.382(T_1 + T_2) + 0.841 T_3). \tag{24}$$

Kelvin mode 2

$$\left(\sigma^{(2)} - 18.1\right)\left(\sigma^{(2)} + 30.3\right) = 0, \quad \text{where } \sigma^{(2)} = 0.595(T_1 + T_2) - 0.541 T_3. \tag{25}$$

Kelvin mode 3

$$\left(\sigma^{(3)} - 21.6\right)\left(\sigma^{(3)} + 9.19\right) = 0, \quad \text{where } \sigma^{(3)} = 0.707(T_1 - T_2). \tag{26}$$

These Kelvin modes are equations of parallel planes in principal stress space (T_1, T_2, T_3) and, along with the maximum principal stress planes, represent the failure bounds in this space. The resulting three-dimensional modal failure envelope is illustrated in Fig. 8(a). The revised paperboard failure envelope displays an intuitively correct strength prediction in the (T_2, T_3) biaxial stress plane (Fig. 8(b)). Though not illustrated, the strength prediction in (T_1, T_3) is identical to that in the (T_2, T_3) plane and is therefore consistent with intuition. The complete three-dimensional failure envelope is the intersection of the Kelvin and complementary mode failure volumes (Fig. 8(a)). There still remains an ambiguity in the

(T_1, T_2) stress plane (Fig. 8(c)). The mode coincidence apparent in Fig. 8(c) is due to the lack of biaxial data for this stress plane.

A comparison of the biaxial modal failure predictions with an appropriate Tsai–Wu envelope (Suhling et al., 1985) is illustrated in Fig. 9. The Tsai–Wu coefficients are those suggested by Suhling et al. (1985): $F_1 = -4.4\mathrm{E}-2\ \mathrm{MPa}^{-1}$, $F_{11} = 2.5\mathrm{E}-3\ \mathrm{MPa}^{-2}$, $F_3 = -3.2\mathrm{E}-2\ \mathrm{MPa}^{-1}$, $F_{33} = 8.9\mathrm{E}-4\ \mathrm{MPa}^{-2}$ and $F_{13} = -3.3\mathrm{E}-4\ \mathrm{MPa}^{-2}$. The stress interaction term suggested by Suhling et al. (1985) was originally determined using the relationship developed by Cowin (1979). Both biaxial strength predictions are superiimposed over a plot of the biaxial strength data.

In a later study, Gunderson et al. (1985) examined a subset of this paperboard data and compared the strength predictions of three 'multiple criteria' failure models. The model of which these authors appear to have obtained the best agreement is one that includes four criteria: a normal compressive strain limit and compressive stress limit, a normal tensile stress limit and what the authors have called a 'combined probability' criterion which allows them to account for the stress interactions in biaxial compressive and tensile stress states. We will consider the work of Gunderson et al. (1985) further in Section 8.

8. Discussion

There are two sets of data which are critical to the determination of the modal failure envelope for any material: the eigenvectors of the elasticity tensor and the bounds or critical values of the Kelvin and complementary modes. The Kelvin modes are planes of constant eigenstress magnitude. The orientations of these planes are dependent upon the eigenvectors of the material's elasticity tensor. It is critical therefore that these eigenvectors be determined as a first step in the analysis. In most cases, the tensor may be measured by either mechanical testing or by ultrasound. Occasionally the method of measurement of the elastic constants is dependent upon the physical limitations of the sample that can be excised from the material. For instance, though highly porous materials such as trabecular bone may display macroscopically homogeneous mechanical behavior, they are in fact inhomogeneous. In such cases a judiciously chosen sample size will provide mechanical measurements that are representative of the material. A sample size of five trabecular lengths is believed to be appropriate for trabecular bone (Harrigan et al., 1988). If the material microstructure can be reconstructed as a computer model then either a finite element model or a reconstruction by rapid polymerization may be used to determine an average elastic behavior and the elasticity tensor deduced from it. Some materials exhibit a visual indication of the principal directions of their material properties, such as the grain of wood. For such a material, a specimen may be shaped in such a way as to accommodate testing in these directions. If the material exhibits no such visual indication then the elasticity tensor may be determined for multiple specimens and the average of the elastic properties may be deduced from the entire set (François, 1995; Cowin and Yang, 1997). Regardless of what method is used, the elasticity tensor is central to the theory because it indicates the symmetry of the material and the orientation of the Kelvin modes.

Once the orientations of the modal failure surface have been established, their position in principal stress space can be determined. For the isotropic case this would be equivalent to establishing the maximum allowable von Mises stress from the available data. However, unlike the isotropic case, an anisotropic material would require a minimum of six pairs of failure data to establish the maximum Kelvin modes. Fortunately, in a principal stress space this number can be reduced to three pairs of failure data since, in many cases, only three modal surfaces can be represented in this stress space. In addition to the Kelvin modes, at least three pairs of failure data are required to establish the maximum complementary modes. This brings the total number of required failure data to six pairs, or twelve test points. Ideally, a succession of three pairs of uniaxial tests followed by three pairs of multiaxial (triaxial) strength tests performed in the direction normal to the Kelvin modes should be performed. These data points would provide a first conservative estimate of the Kelvin mode failure bounds. Following these six triaxial tests, six additional mechanical tests could be performed in the direction least likely to intercept a Kelvin mode failure surface (the directions of which would be apparent from the initial failure envelope). Each consecutive set of data would produce successively less conservative failure envelopes. The failure envelope would be presumed to be completely defined when a sufficient number of failure data points could be uniquely attributed to each failure mode.

Unfortunately, a reliable procedure for triaxial testing of anisotropic materials is by no means established. Multiaxial tests are difficult to reproduce accurately even for isotropic materials so it is unlikely that this type of data would be available for anisotropic materials in the near future. It is more reasonable to expect that biaxial data may be available. It is possible to determine the mode bounds from a set of biaxial strength data in mutually orthogonal stress planes. With such data, the position of the failure modes is simply the outermost envelope, one which includes all of the strength data. If the theory is applicable, the data should closely match the envelope created by the Kelvin and complementary modes. To illustrate this method the paperboard data of Suhling et al. (1985) used by Biegler and Mehrabadi (1993) was re-examined (Fig. 3). In this case, however, the entire set of biaxial stress data is employed to produce the mode bounds and the resulting three-dimensional failure envelope reproduced (Fig. 8(a)). The fit of this modified modal theory to the (T_2–T_3) biaxial strength experimental data is unusually good (Fig. 8(b)). The biaxial compressive strength is consistent with the intersection of a Kelvin mode 1 and a complementary mode. The combined T_2 (tensile) and T_3 (compressive) strength data also appear to be consistent with the intersection of a Kelvin mode 2 and a complementary mode. The inclusion of the biaxial data in the evaluation of the modal envelope appears to have been beneficial. The resulting fit of the modal theory to the biaxial T_2 (tensile) and T_3 (compressive) strength data (Fig. 8(b)) is even better than that obtained by Biegler and Mehrabadi (1993) (Fig. 3). Although not plotted in this study, the (T_1–T_3) trace is identical to that of the (T_3–T_2) biaxial stress plane. The lack of biaxial transverse strength is apparent in the (T_1–T_2) biaxial trace (Fig. 8(c)). If such data were available, the trace in this plane may have a somewhat different appearance.

There are some immediately apparent circumstances that preclude the applicability of this failure theory. The first would be a concavity in the failure envelope. Such features are sometimes apparent with materials that exhibit buckling in biaxial compression (Rowlands, 1985). The modal theory presented in this study produces a convex envelope for all symmetries including isotropy (von Mises criteria). In fact, all of the anisotropic failure criteria suggested so far produce convex envelopes and would obviously not accommodate such a feature. Clearly, specially tailored empirical functions would have to be developed to describe the strength behavior of these materials.

The close connection between the elasticity and the Kelvin mode strength predictions requires not only that the material be linearly elastic within the Kelvin and complementary failure bounds but that the anisotropy remain constant within it as well. This is not always the case, particularly for aggregates such as concrete, rock and soils.

This modal failure theory is not the first attempt to correlate the failure stress of an anisotropic material with a failure mechanism but the first to extend von Mises plasticity theory to anisotropic materials. The results presented in this study are encouraging but not conclusive. The paperboard biaxial modal failure envelope is in excellent agreement with experiment and appears to account for the discontinuities in the strength distribution reported by Suhling et al. (1985). However, no association of a failure mechanism with these strength data was reported. So it is impossible to tell if the failure data associated with a particular Kelvin or complementary mode can also be attributed to a particular failure mechanism. It is tempting to correlate the strength predictions of this modal failure theory with that of Gunderson et al. (1985). Both approaches fit the biaxial data reasonable well where data are available. Where there are no available data (e.g. for T_1 slightly compressive and T_3 nearly tensile nearly at the maximum uniaxial limit) both failure envelopes exhibit roughly similar slopes. However, it is inappropriate to use one strength predictor to validate another. Since Gunderson et al. (1985) do not correlate the failure modes in their study with direct observation of mechanism of failure, any similarity between the approach used in their study and this one validates neither.

The most significant remaining shortcoming of this modal failure theory is the likely requirement of an undetermined number of multiaxial tests to completely define the failure envelope. Indeed, because the Kelvin and complementary modes are not orthogonal to each other in stress space, it is likely that more than the minimum twelve mechanical tests would be required. An appropriate sequence of tests might be a set of three (positive and negative) pairs of uniaxial tests in the three principal directions of the material followed by three pairs of triaxial tests in the direction of the modal stresses. The three modal and three principal stress magnitudes would then be tabulated and the maximum and minimum values identified for each column. If only one mode (Kelvin or complementary) has reached an extremum at a particular strength datum, then that mode is presumed to have been uniquely determined and may safely be ignored in any subsequent iteration. If on other hand, a strength datum can be attributed to the extremum of two or more modes, then this point represents an ambiguity which must be resolved by additional mechanical tests. Fortunately, because all successive failure envelopes are necessarily convex, they also represent progressively less conservative failure predictions.

As empirical curves are not obtained from theories, they fit the experimental data as closely as possible regardless of how the data were obtained or what measure they are meant to represent. In this sense, criteria such as the Tsai–Wu and Hill criterion represent implicit predictors of strength. These criteria are based on the belief that, if two similar stress states represent two failure points, then some in-between stress state must also be a similar failure point. When the Tsai–Wu criterion is combined with a predictor of its coefficients, the resulting criterion is more than an empirical relation. For instance, the paperboard Tsai–Wu criterion suggested by Suhling et al. (1985) for paperboard is such an example because the stress interaction terms F_{ij} are functions of the uniaxial and shear strengths (Cowin, 1979). The biaxial strength disitribution is thus predicted from uniaxial and shear strength. On the other hand, the modal energy failure criterion of Biegler and Mehrabadi (1993) is a genuine theory. This theory postulates that an anisotropic material fails when the energy in any one of its six independent modes has reached some critical value. The theory is entirely analogous to the theory of failure by maximum distortional or dilatational energy for isotropic materials. The superposition of the complementary modes is analogous to that of a maximum principal stress criterion over the distortional and dilatational energy criterion. The necessity of this additional set of failure modes became apparent after the observation that the modal energy criterion alone could not account for the different compressive and tensile strengths of paperboard.

The implied correlation of the principal directions of the strength envelope with those of the elasticity tensor makes this theory unique. Organic materials such as wood and bone have demonstrated connections between their elasticity and strength, so that a failure theory such as this one may be particularly well suited for them.

9. Conclusion

The failure criterion of Biegler and Mehrabadi (1993) was introduced and a modified procedure for the determination of the modal failure envelope was proposed. This procedure involves the determination of the maximum and minimum modal stress magnitudes (Kelvin modes) rather than the modal energies. It was further shown that the Kelvin mode theory provides an incomplete description of the failure of some materials, but that this weakness can be addressed by the introduction of a set of complementary modes. This combined modal theory was applied to the tetragonal paperboard example used by Biegler and Mehrabadi (1993). The resulting failure envelope for this example was shown to produce intuitively correct failure predictions. The Kelvin and complementary principal stress modes appear to better account for the paperboard strength distribution than the best fitting Tsai–Wu.

The value of this work lies in the nature of its strength predictions. The Kelvin mode criterion is based on the premise that an anisotropic material will fail when the energy in any one of up to six independent modes has reached a critical value (Biegler and Mehrabadi, 1993). Because it is based on this premise the Kelvin mode criterion is an explicit predictor of the strength rather than an implicit one such as the Tsai–Wu, Hill and other criteria. The Kelvin mode criterion is therefore a genuine theory of the multiaxial strength of any anisotropic elastic material and appears to be a viable and practical alternative to the Tsai–Wu and other current anisotropic strength criteria.

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